

Representation of Fractional Factorial Design in Terms of (0,1)-Matrices

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SUMMARY

Let T denote a main effect plan for the s^n factorial with N assemblies, that is T is an $N \times n$ matrix with elements from the $\{0, 1, \dots, s-1\}$. Denote by T_0, T_1, \dots, T_{s-1} the $N \times n$ incidence matrices of $0, 1, \dots, s-1$ respectively, so that $T = \sum_i T_i$ and $\sum_i T_i = J_{N \times n}$. Using the Helmert polynomials to define single degree of freedom main effect contrasts we write $E\{\underline{y}\} = X\beta$, where X is the design matrix corresponding to T . A transformation G is obtained for which $XG = X^* = [\underline{1} \vdots T_1 \vdots \dots \vdots T_{s-1}]$ thus giving a representation for the design matrix directly in terms of the (0,1)-incidence matrices. It is shown that $|G| = (s!)^{-n}(-1)^{(s-1)n}$ and $|X'X| = (s!)^{2n}|X^*'X^*|$. If T is a saturated main effect plan, then $|X| = (s!)^n|X^*|$. Thus the determinant of the information matrix is directly expressible in terms of the determinant of a (0,1)-matrix. These results are extended to include the general asymmetrical factorial $\prod s_i$. Upper bounds are obtained for the determinant values of X^* when X^* is square and in general for $X^*'X^*$. One important aspect of this representation is that the construction of main effect plans and an assessment of their goodness via the determinant criteria can be studied directly in terms of (0,1) matrices. An extension to include interaction terms for the s^n factorial where s is a prime or prime power is given.

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1. INTRODUCTION

In a factorial experiment involving n factors with the i th factor at s_i levels $i = 1, 2, \dots, n$, the total number of treatment combinations is $\prod_{i=1}^n s_i$. A design is usually represented as an $N \times n$ matrix T whose N rows denote the particular treatment combinations and whose n columns correspond to the levels of the n factors. The elements of a column of T corresponding to a factor at s_i levels are integers from $\{0, 1, \dots, s_i - 1\}$, $i = 1, 2, \dots, n$, to denote the s_i levels.

In the analysis, the matrix T is almost universally replaced by a matrix $X_{N \times v}$ which reflects the v single degree of freedom parametric contrasts in the parametric vector $\underline{\beta}_{v \times 1}$ from the usual regression equation $E\{\underline{y}\} = X \underline{\beta}$ and $\text{Cov}\{\underline{y}\} = \sigma^2 I_N$. The normal equations are $X'X\hat{\underline{\beta}} = X'\underline{y}$, and solutions to the normal equations provide best linear unbiased estimates of estimable functions of the parameters in $\underline{\beta}$. The matrix $X'X$ is called the information matrix of the design T , and if $X'X$ is nonsingular the variance-covariance matrix of $\underline{\beta}$ is proportional to $(X'X)^{-1}$. Most criteria of goodness of a design depend upon some function of $(X'X)^{-1}$, as for example, the determinant, trace, and maximum root criteria. If $X'X$ is singular we consider a conditional inverse $(X'X)^{-}$ and restrict to estimable functions of the β 's.

Since the matrix X is obtained directly from the matrix T , all of the information concerning the goodness of the design (in terms of some function of $(X'X)^{-1}$) is contained within T . Thus for the purpose of constructing good designs and the comparison of designs, the simplest and most direct representation of this property in terms of T itself would be useful.

Raktoe and Federer [1970] obtained such a representation directly in terms of the $(0,1)$ matrix $(\underline{1}; T)$ for main effect plans for the 2^n factorial,

where $\underline{1}$ denotes a vector with every element unity. In Section 2 of this paper we present a similar representation for the $\prod_{i=1}^n s_i$ factorial, and we represent $|(X'X)^{-1}|$ directly in terms of this representation. In the third section an upper bound on the $|X'X|$ is obtained for both the symmetric and asymmetric factorials, and the minimum nonzero value of this determinant is indicated.

The importance of the representation presented lies in the insight that may be gained toward the construction of fractional factorial plans and the assessment of their goodness via the determinant criteria.

2. REPRESENTATION OF MAIN EFFECT PLANS IN TERMS OF (0,1)-MATRICES

Consider first the s^n symmetrical factorial and the corresponding main effect plans for estimating the $v = 1 + n(s-1)$ mean and main effects under the assumption that all two-factor and higher-factor interactions are zero. Let $T_{N \times n}$ be an $N \times n$ matrix, $N \geq v$, with elements from the set $\{0, 1, \dots, s-1\}$ denoting such a main effect plan. Let T_i , $i = 0, 1, \dots, s-1$, be the $N \times n$ incidence matrix of element i in T . That is, an element of T_i is one or zero as the corresponding element of T is i or not. Then,

$$\sum_{i=0}^{s-1} T_i = J_{N \times n}, \quad \text{and} \quad T = \sum_{i=0}^{s-1} iT_i. \quad (2.1)$$

Typically main effects are defined in terms of a set of orthogonal polynomials. For convenience, we shall use the Helmert polynomials,

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & & 0 \\ 1 & 1 & -2 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \dots & -(s-1) \end{bmatrix} \quad (2.2)$$

even though any set of orthogonal polynomials may be used. Then, if \underline{y} denotes an $N \times 1$ observation vector corresponding to T , let

$$E[\underline{y}] = X\underline{\beta} \quad \text{and} \quad \text{Cov}(\underline{y}) = \sigma^2 I_N,$$

where $\underline{\beta} = (\mu, \beta_1, \dots, \beta_n, \beta_1^2, \dots, \beta_n^2, \beta_1^3, \dots, \beta_n^{s-1})$ denotes the $v \times 1$ parameter vector of single degree of freedom contrasts as derived from Helmert polynomials and where X is the design matrix. The design matrix may be written in terms of the T_i as

$$X = [\underline{1} \ ; \ T_0 - T_1 \ ; \ T_0 + T_1 - 2T_2 \ ; \ \dots \ ; \ \sum_{i=0}^{s-2} T_i - (s-1)T_{s-1}]. \quad (2.3)$$

In theorems 2.1 and 2.2 we transform the design matrix X for a main effect plan from the s^n factorial into a $(0,1)$ -matrix. The results are extended in theorems 2.3 and 2.4 for the general $\prod_{i=1}^n s_i$ asymmetrical factorial. The importance of these results centers around the facts that (i) considerable theory is available on the construction of main effect plans for the 2^n factorial and on the values of the determinants of $(0,1)$ -matrices, (ii) this theory can now be applied to the construction and to the consideration of optimality of main effect plans from the general factorial, and (iii) these results extend the results of Raktoe and Federer [1970] for the 2^n factorial to the general symmetrical and asymmetrical factorials.

Consider the column operations on X resulting from postmultiplying by a matrix G as follows:

$$XG = X^*, \quad (2.4)$$

where G is the following $v \times v$ matrix:

$$G = \begin{bmatrix} 1 & \frac{1}{s} \underline{1}' & \frac{1}{s} \underline{1}' & \frac{1}{s} \underline{1}' & \dots & \frac{1}{s} \underline{1}' \\ 0 & -\frac{1}{2} I_n & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2(3)} I_n & -\frac{1}{3} I_n & 0 & \dots & 0 \\ 0 & \frac{1}{3(4)} I_n & \frac{1}{3(4)} I_n & -\frac{1}{4} I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{s(s-1)} I_n & \frac{1}{s(s-1)} I_n & \frac{1}{s(s-1)} I_n & \dots & -\frac{1}{s} I_n \end{bmatrix} \quad (2.5)$$

The following results can be verified by direct calculation.

Theorem 2.1. Under the transformation $XG = X^*$, we have

- (a) $X^* = [\underline{1} : T_1 : T_2 : \cdots : T_{s-1}]$, that is, the transformed design matrix is a $(0,1)$ - matrix composed of incidence matrices T_i , $i = 1, 2, \dots, s-1$;
- (b) $|G| = (s!)^{-n} (-1)^{n(s-1)}$;
- (c) $G^{-1} = \begin{bmatrix} 1 & \underline{1}' & \underline{1}' & \underline{1}' & \dots & \underline{1}' & \underline{1}' \\ \underline{0} & -2I_n & 0 & 0 & 0 & 0 & 0 \\ \underline{0} & -I_n & -3I_n & 0 & 0 & 0 & 0 \\ \underline{0} & -I_n & -I_n & -4I_n & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{0} & -I_n & -I_n & -I_n & -(s-1)I_n & 0 & 0 \\ \underline{0} & -I_n & -I_n & -I_n & \dots & -I_n & -sI_n \end{bmatrix}$;
- (d) $X = X^*G^{-1}$; and
- (e) $|X^*X| = (s!)^{2n} |X^*X^*|$.

Theorem 2.2 If $N = v$ and the design is a saturated main effect plan, the determinant of the resulting square matrix may be expressed as:

$$|X| = (-1)^{n(s-1)} (s!)^n |X^*X^*|.$$

The proof follows directly from parts (a), (b), and (d) of theorem 2.1.

Example 2.1 Consider the saturated main effect plan for the 3^4 factorial derived from the following pair of orthogonal latin squares:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Then,

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The determinant of X is $|X| = (3!)^4 |[\underline{1} : T_1 : T_2]| = (3!)^4 (27) = (3!)^4 |X^*|$. Since T is an orthogonal array, 27 is the maximum value possible for $|X^*|$ of a 3^4 saturated main effect plan.

The effect of the structure constraints (2.1) on T_0 , T_1 , and T_2 is apparent from example 2.1. The value 27 is far below the maximum value of the determinant of a $(0,1)$ -matrix of size 9 with a leading column of ones. In fact, for such matrices, Anderson and Federer [1974] obtained the following values: all integers $\leq 33, 36, 40, 44, 48$, and 56, with no assurance that all values have been obtained or that 56 is the maximum value. Thus the largest possible value for the determinant of form $|\underline{1} : T_1 : T_2|$ where T_1 and T_2 satisfy (2.1) is an intermediate value among all possible values.

The results for the s^n factorial are now extended to the general asymmetrical factorial $s_1 \times s_2 \times \dots \times s_n = \prod_{i=1}^n s_i$. Since factors have different numbers of levels, it will be convenient to consider the representation individually for each factor. Thus let the i th column of T corresponding to the i th factor be denoted by \underline{d}_i so that $T = [\underline{d}_1 \ \underline{d}_2 \ \dots \ \underline{d}_n]$. Since F_i has s_i levels here denoted by $0, 1, \dots, s_i - 1$, the column \underline{d}_i contains these symbols. Let the incidence matrix of the level j in the i th column \underline{d}_i be denoted by $\underline{d}_i(j)$. The equations (2.1) are thus expressed

for each i as

$$\sum_{j=0}^{s_i-1} \underline{d}_i(j) = \underline{1} \quad \text{and} \quad \underline{d}_i = \sum_{j=0}^{s_i-1} j \underline{d}_i(j). \quad (2.6)$$

The s_i-1 columns in the X matrix corresponding to the factor F_i are given by

$$[\underline{d}_i(0)-\underline{d}_i(1) : \underline{d}_i(0)+\underline{d}_i(1)-2\underline{d}_i(2) : \dots : \sum_{j=0}^{s_i-2} \underline{d}_i(j) - (s_i-1)\underline{d}_i(s_i-1)], \quad (2.7)$$

and we let

$$Z_i^* = [\underline{d}_i(1) \quad \underline{d}_i(2) \quad \dots \quad \underline{d}_i(s_i-1)]. \quad (2.8)$$

Finally, let

$$G_i = \begin{bmatrix} \underline{1} & \underline{1}'/s_i \\ \underline{0} & H_i \end{bmatrix} = \begin{bmatrix} 1 & 1/s_i & 1/s_i & \dots & 1/s_i & 1/s_i \\ 0 & -1/2 & 0 & \dots & 0 & 0 \\ 0 & 1/2(3) & -1/3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1/s_i(s_i-1) & 1/s_i(s_i-1) & \dots & 1/s_i(s_i-1)-1/s_i & \dots \end{bmatrix}$$

denote the matrix G of (2.5) written only for the i th factor at s_i levels, and

for all n factors let

$$G = \begin{bmatrix} 1 & \underline{1}'/s_1 & \underline{1}'/s_2 & \dots & \underline{1}'/s_n \\ \underline{0} & H_1 & 0 & \dots & 0 \\ \underline{0} & 0 & H_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & 0 & 0 & \dots & H_n \end{bmatrix} \quad (2.9)$$

It can be ascertained that if X is the design matrix for a main effect plan from the asymmetrical factorial that the following two theorems hold:

Theorem 2.3(a) $XG=X^* = [\underline{1} : Z_1^* : Z_2^* : \dots : Z_n^*]$, where X^* is a $(0,1)$ -matrix with leading column $\underline{1}$;

(b) $|G| = \prod_{i=1}^n (s_i!)^{-1}$ and

(c) $|X \tilde{X}| = \prod_{i=1}^n (s_i!)^2 |X^* \tilde{X}^*|$

Theorem 2.4. If T is a saturated main effect plan for the asymmetrical $\prod_{i=1}^n s_i$ factorial, then

$$|X| = \prod_{i=1}^n (s_i!) |X^*|.$$

Note that the columns of X^* are ordered so that all (s_i-1) columns corresponding to the i th factor appear together. This ordering is possible and sometimes preferred for the symmetric case also. The conditions (2.6) are actually structure constraints on X^* . For example, each row of Z_i^* , $i = 1, 2, \dots, n$, can have at most one value of one, hence the inner product of any two columns of Z_i^* is zero.

Example 2.2 A design for a $2^2 \times 3 \times 4$ factorial in eight runs and its corresponding (0,1) representation are given as

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix} \quad X^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 000 \\ 1 & 0 & 0 & 1 & 0 & 100 \\ 1 & 0 & 1 & 0 & 1 & 010 \\ 1 & 0 & 1 & 0 & 0 & 001 \\ 1 & 1 & 1 & 1 & 0 & 000 \\ 1 & 1 & 1 & 0 & 1 & 100 \\ 1 & 1 & 0 & 0 & 0 & 010 \\ 1 & 1 & 0 & 1 & 0 & 001 \end{bmatrix}.$$

It is easily shown that $|X^*| = 6$ so from Theorem 2.4 $|X| = 2!2!3!4! \cdot 6$.

3. BOUNDS ON THE DETERMINANTS OF NONSINGULAR DESIGN MATRICES

The transformation from X to X^* provides a simple proof that the determinant of $X'X$ is invariant to any change of level designation for any factor. Any permutation of the non zero levels results only in a corresponding permutation of columns in X^* which of course does not change the value of the determinant. Likewise, any non zero level may be interchanged with the zero level for any specified factor. The corresponding change in X^* is a linear combination of the first column of all ones and the columns of T_1, T_2, \dots, T_{s-1} corresponding to that factor. Again this does not change the determinant. This invariance property is a well known result, see for example Paik and Federer [1970] and Srivastava, Raktoe, and Pesotan [1971], but the representation in terms of (0,1) matrices makes it more apparent.

Let σ_i^k , $k = 0, 1, 2, \dots, s-1$ denote the number of treatment combinations of a fraction T which contain the i^{th} factor at level k. Then, $\sum_{k=0}^{s-1} \sigma_i^k = N$ for each i. In any discussion involving the determinant of X, or of X^* , we may, without loss of generality, assume that $\sigma_i^0 \geq \sigma_i^1 \geq \sigma_i^2 \geq \dots \geq \sigma_i^{s-1}$ for each i because of the invariance property.

Raktoe and Federer [1970] obtained the following bound on $\|X^*\|$ using Hadamard's theorem:

$$\|X^*\| \leq (n+1)^{(n+1)/2} 2^{-n} \quad (3.1)$$

Since $|X^*|$ must be an integer, we take the integer part of the right hand side of (3.1) as the upper bound. We now obtain a generalization of their result for X^* matrices, and consequently X matrices, for saturated main effect plans from the symmetrical s^n factorial.

Theorem 3.1. Let T be a saturated main effect plan for the s^n factorial with $N = n(s-1) + 1$. If $X^* = [\underline{1} : T_1 : T_2 : \dots : T_{s-1}]$, then

$$|X^*| \leq \text{integer part of } N^{N/2} s^{-sn/2}. \quad (3.2)$$

When $s = 2$, this reduces to equation (3.1).

Proof: From theorem 2.2,

$$\|X^*\| = (s!)^{-n} \|X\| = (s!)^{-n} |X'X|^{1/2}. \quad (3.3)$$

From Hadamard's determinant theorem, we know that $|X'X|$ is less than or equal to the product of its diagonal elements with equality only if $X'X$ is a diagonal matrix. Using equations (2.5) and (3.3), we obtain:

$$\|X^*\|^2 \leq (s!)^{-2n} N \prod_{i=1}^n \prod_{k=1}^{s-1} (\sigma_i^0 + \sigma_i^1 + \dots + k^2 \sigma_i^k), \quad (3.4)$$

where we take $\sigma_i^0 \geq \sigma_i^1 \geq \dots \geq \sigma_i^{s-1}$ for each i.

Expression (3.4) will be maximized whenever each of the interior products is maximized; thus, we need only consider

$$\prod_{k=1}^{s-1} (\sigma_i^0 + \sigma_i^1 + \dots + \sigma_i^{k-1} + k^2 \sigma_i^k). \quad (3.5)$$

Next introduce the Lagrange multiplier corresponding to the constraint $\sum_{k=0}^{s-1} \sigma_i^k - N = 0$, and take derivatives with respect to σ_i^{s-2} and σ_i^{s-1} .

Equating these two derivatives, we obtain an expression in σ_i^{s-2} and σ_i^{s-1} as follows:

$$\sigma_i^{s-2} = \frac{(s-2)^2 + 1}{(s-1)(s-3)} \sigma_i^{s-1} - \frac{2N}{s(s-1)(s-3)}. \quad (3.6)$$

The equations are satisfied when $\sigma_i^{s-2} = \sigma_i^{s-1} = N/s$. We may assume that $\sigma_i^{s-2} \geq \sigma_i^{s-1}$ and that $\sigma_i^{s-1} \leq N/s$ from the ordering previously described.

Whenever $\sigma_i^{s-1} < N/s$, we have $\sigma_i^{s-2} < \sigma_i^{s-1}$ from (3.6). Hence, it follows that $\sigma_i^{s-1} = \sigma_i^{s-2} = N/s$ and since the smallest of the σ_i^k equals N/s and since their total is N , we have

$$\sigma_i^0 = \sigma_i^1 = \sigma_i^2 = \dots = \sigma_i^{s-1} = N/s, \quad i = 1, 2, \dots, n. \quad (3.7)$$

Thus,

$$\begin{aligned} \|X^*\| &\leq (s!)^{-n} \{N^N \prod_{k=1}^{s-1} k(k+1) s^{-N+1}\}^{1/2} \\ &= (s!)^{-n} N^{N/2} [(s-1)!s!]^{n/2} s^{-n(s-1)/2} \\ &= N^{N/2} s^{-ns/2}. \end{aligned}$$

Corollary 3.1 Let T be a main effect plan for an s^n factorial experiment with $N \geq n(s-1)+1$. Then

$$|X^* X^*| \leq \text{integer part of } N^{n(s-1)+1} s^{-ns}. \quad (3.8)$$

Proof The proof of theorem 3.1 uses $|X'X|$ and the essential steps do not depend on $N = n(s-1)+1$. Hence the proof is complete.

Example 3.1 Consider a set of t orthogonal latin squares of order s . This set may be regarded as an orthogonal main effect plan for the s^{t+2} factorial with $N=s^2$. If $t = s-1$, which is possible whenever s is a prime or prime power, the set forms a saturated main effect plan. The $s^2 \times (1+(t+2)(s-1))$ matrix X^* is given by $X^* = [\underline{1} \quad T_1 \quad T_2 \dots T_{s-1}]$ where

$T_i' T_i = (s-1)I + J$ and $T_i' T_j = J-I$ $i \neq j$. Thus

$$X^* ' X^* = \begin{bmatrix} s^2 & s\underline{1}' & s\underline{1}' & \dots & s\underline{1}' \\ s\underline{1} & sI+(J-I) & J-I & \dots & J-I \\ s\underline{1} & J-I & sI+(J-I) & \dots & J-I \\ \vdots & & & & \\ s\underline{1} & J-I & J-I & \dots & sI+(J-I) \end{bmatrix}. \quad (3.9)$$

The determinant of $X^* ' X^*$ is $|X^* ' X^*| = s^{(t+2)(s-2)+2}$ which equals the bound given in corollary (3.1). When $t = s-1$, the design is saturated. Then $|X^* ' X^*| = s^{s(s-1)}$ and $|X^*| = s^{s(s-1)/2}$ which equals the bound given in theorem 3.1.

Example 3.2 Suppose T is an orthogonal array of size N , n constraints, s levels, of strength 2, and index λ denoted by $(N, n, s, 2)$. That is, T is a fraction for an s^n factorial in N runs such that for any pair of factors each of the s^2 possible combinations of levels occurs exactly λ times. Clearly $N = \lambda s^2$ and the matrix $X^* ' X^*$ for this fraction is exactly λ times the matrix (3.9) in example 3.1. The determinant of $X^* ' X^*$ also attains the upper bound given in corollary 3.1.

The upper bound for the general asymmetrical factorial may be proved in a similar manner since the maximization is essentially for a single factor at a time. The results are contained in the following theorem and corollary.

Theorem 3.2 Let T be a saturated main effect plan for a general asymmetrical $\prod_{i=1}^n s_i$ factorial with $N = 1 + \sum_{i=1}^n (s_i - 1)$ runs and let X^* be the $(0,1)$ -matrix of theorems 2.3(a) and 2.4. Then

$$||X^*|| \leq \text{integer part of } N^{N/2} \prod_{i=1}^n s_i^{-s_i/2}.$$

Corollary 3.2 Let T be a main effect plan for a general asymmetrical $\prod_{i=1}^n s_i$ factorial with $N \geq 1 + \sum_{i=1}^n (s_i - 1)$ runs and let X^* be the corresponding $(0,1)$ -matrix representation. Then

$$|X^* ' X^*| \leq \text{integer part of } N^v \prod_{i=1}^n s_i^{-s_i}, \quad v = 1 + \sum_{i=1}^n (s_i - 1).$$

Example 3.3. Proportional frequency designs and sets of orthogonal F squares as discussed by Hedayat and Seiden [1970] provide examples of orthogonal main effect plans for the asymmetric factorial. It can be shown that the structure of X^*X^* for these designs is similar to (3.9).

Theorem 3.3. The class of saturated main effect designs for the $\prod_{i=1}^n s_i$ factorial contains designs for which $|X^*| = 1$. That is the minimum possible nonzero value is always attainable.

Proof. The familiar "one at a time design" has a (0,1) representation as

$$X^* = \begin{bmatrix} 1 & 00\dots 0 \\ \frac{1}{v-1} & I_{v-1} \end{bmatrix}, \quad v-1 = \sum_{i=1}^n (s_i - 1),$$

whose determinant is clearly one. The proof is complete since one design is exhibited for every case.

Corollary 3.3. If T is a saturated main effect plan for the $\prod_{i=1}^n s_i$ factorial, the minimum possible value of $|X'X|$ is $\prod_{i=1}^n (s_i - 1)!$ and this value is always attainable. Thus for any saturated design the $|X'X|$ is a multiple of this minimum value.

Proof. This follows directly from theorems 2.3 and 3.3.

Thus for saturated main effect plans the smallest value of the determinant of X, or X^* , can always be attained. The upper bound on the determinant of X, or X^* , will be attained whenever an orthogonal saturated main effect design with equal numbers of repetitions on the levels of each factor is obtained. In the 3^n series, for example, this will occur with $n = 4$ and $N = 9$ yielding $|X^*| = 3^3$; the next orthogonal saturated main effect plan occurs for $n = 13$ and $N = 27$ yielding $|X^*| = 3^{21}$. In cases where an orthogonal design does not exist the upper bound will not be attained.

4. ON THE CONSTRUCTION OF MAIN EFFECT PLANS

The construction of main effect plans for the symmetric and asymmetric factorial is now directly related to constructions of $(0,1)$ matrices with certain constraints on the columns. Thus the body of knowledge and developed theory of $(0,1)$ matrices can be directly brought to the construction of main effect plans. In this section we illustrate this with a few types of constructions. Recall from (2.1) that for a factor at s levels there must be a corresponding set of $(s-1)$ columns in X^* with all pairwise inner products zero and among these columns at least one row must be all zero.

Example 4.1. Circulant Matrix Construct. Let C_{2n} be a $2n \times 2n$ circulant matrix whose first row contains ones and zeros such that the i^{th} and $(n+i)^{\text{th}}$ coordinates are not both one. The remaining rows of C are of course just cyclic permutations of the first row. Let X^* be

$$X^* = \begin{bmatrix} 1 & 0 & 1 \times 2n \\ 1_{2n \times 1} & C_{2n \times 2n} \end{bmatrix}.$$

This X^* matrix is appropriate for a 3^n saturated main effect plan, and since the theory of circulants is well known the determinant is easy to evaluate. To illustrate we list the first row of a suitable C matrix for the 3^n factorial with $n = 3, 4, 5, 6$ and 7 and give the corresponding determinant of X^* .

n	First row of C	Det. of X^*
3	(1 0 1 0 0 0)	4
4	(1 1 0 1 0 0 0 0)	27
5	(1 1 1 0 1 0 . . . 0)	88
6	(1 1 1 0 1 0 . . . 0)	208
7	(1 1 0 1 1 0 1 0 . . . 0)	420

A similar construction for the s^n factorial would require a $(s-1)n \times (s-1)n$ circulant matrix with at most one one in the $i, n+i, 2n+i, \dots, (s-1)n+i$ columns $i = 1, 2, \dots, n$.

Example 4.2. Sum Composition - Let T_1, T_2, \dots, T_{s-1} be $n \times n$ matrices of ones and zeros, and let

$$X^* = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \dots & \underline{0} \\ \underline{1} & T_1 & 0 & \dots & 0 \\ \underline{1} & 0 & T_2 & \dots & 0 \\ \vdots & & & & \\ \underline{1} & 0 & 0 & \dots & T_{s-1} \end{bmatrix}.$$

Clearly $|X^*| = \prod_{i=1}^{s-1} |T_i|$, and the corresponding design is

$$T = \begin{bmatrix} \underline{0} & 1 \times n \\ T_1 \\ 2T_2 \\ 3T_3 \\ \vdots \\ (s-1)T_{s-1} \end{bmatrix}.$$

Anderson and Federer [1974] considered possible values for the determinant of (0,1)-matrices and used ten methods of construction to obtain many of the possible values. Here we present all possible determinant values attainable by the above method of construction for saturated main effect plans from the 3^n series for $n = 3, 4, 5, 6$, and 7 .

$$n = 3: 6^3[0, 1, 2, 4]$$

$$n = 4: 6^4[0, 1, 2, 3, 6, 9]$$

$$n = 5: 6^5[0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 20, 25] = 6^5 [0, 1, 2, 3, 4, 5]^2$$

$$n = 6: 6^6[0, 1, 2, \dots, 9]^2 = 6^6[\text{all possible products of integers } 0, 1, \dots, 9]$$

$$n = 7: 6^7[\text{all integers } \leq 18, 20, 24, 32]^2$$

where the integers within a square bracket represent possible values for the determinant of X^* .

It should be noted that this construction is restrictive and does not provide all possible values of $|X|$. For example, for $n = 3$, and for another construction, it is possible to obtain a design for which $X = 6^3(3)$ and which is not obtained via the above construction. Even though this method

of construction gave the largest value obtained for $n = 3$, it is expected that this will hold for larger n . When $n = 4$, the orthogonal saturated design in example 2.1 yields a design for which $|X| = 6^4(27)$, which is three times larger than the largest value obtained from this sum composition. The spectrum of possible values or even the largest possible value of $|X|$ is unknown at present. The transformation of X to X^* , i.e., a $(0,1)$ -matrix, is considered to be one step toward the resolution of these problems.

Example 4.3. The construction of example 4.2 can be extended to the general main effect plans. Let T_1, T_2, \dots, T_{s-1} be $(0,1)$ -matrices of order $N_1 \times n, N_2 \times n, \dots, N_{s-1} \times n$, respectively, for $N_i \geq n$, such that $[0 \ T'_1]$ could be regarded as a main effect plan for the 2^n factorial with $N_1 + 1$ runs. Now, consider the following design for the s^n factorial with $N = 1 + \sum_{i=1}^{s-1} N_i$ runs:

$$T = [0 \ T'_1 \ 2T'_2 \ \dots \ (s-1)T'_{s-1}]'$$

Then,

$$X^* = \begin{bmatrix} 1 & 0' & 0' & \dots & 0' \\ \underline{1} & T_1 & 0 & & 0 \\ \underline{1} & 0 & T_2 & & 0 \\ \vdots & & & & \vdots \\ \underline{1} & 0 & 0 & \dots & T_{s-1} \end{bmatrix},$$

and

$$X^* ' X^* = \begin{bmatrix} N & \underline{1}' T_1 & \underline{1}' T_2 & \dots & \underline{1}' T_{s-1} \\ T'_1 \underline{1} & T'_1 T_1 & 0 & & 0 \\ T'_2 \underline{1} & 0 & T'_2 T_2 & & 0 \\ \vdots & & & & \vdots \\ T'_{s-1} \underline{1} & 0 & 0 & \dots & T'_{s-1} T_{s-1} \end{bmatrix}.$$

Given the T_i , $i = 1, 2, \dots, s-1$, it is a relatively simple matter to compute $|X^* ' X^*|$.

To conclude, we suggest one additional construction for main effect plans from the s^n factorial. This method makes use of a $(0,1)$ -matrix T and its complement $(J-T)$ and by arranging these matrices to satisfy constraints (2.1) and (2.2). We

illustrate the procedure for the 3^n , the 4^n series, and then for the s^n series.

Example 4.4. Let T be an $N \times n$ $(0,1)$ -matrix of full rank with $N \geq n$.

For the 3^n series, consider the plan defined by

$$T_1 = \begin{bmatrix} T \\ J - T \\ 0 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 0 \\ T \\ J - T \end{bmatrix}$$

with $3N$ runs. Each of the three levels of each factor occurs N times. For this design,

$$X^* ' X^* = \begin{bmatrix} 3N & \underline{N1}' & \underline{N1}' \\ \underline{N1} & T' T + (J-T)'(J-T) & (J-T)' T \\ \underline{N1} & T' (J-T) & T' T + (J-T)'(J-T) \end{bmatrix}.$$

If T itself is a structured matrix, then $X^* ' X^*$ has a simple structure. For example, if $T = I_n$, then

$$X^* ' X^* = \begin{bmatrix} 3n & \underline{n1}' & \underline{n1}' \\ \underline{n1} & nI + (n-2)(J-I) & (J-I) \\ \underline{n1} & (J-I) & nI + (n-s)(J-I) \end{bmatrix}$$

and

$$|X^* ' X^*| = 3^{n-1} n(n^2 - 3n + 3)^2.$$

For the 4^n series, the construction is given by

$$T_1 = \begin{bmatrix} T \\ J - T \\ 0 \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 \\ T \\ J - T \\ 0 \end{bmatrix}, \text{ and } T_3 = \begin{bmatrix} 0 \\ 0 \\ T \\ J - T \end{bmatrix}$$

In general, for the s^n factorial we let T_i , $i = 1, 2, \dots, s-1$, be $sN \times n$ matrices whose i^{th} and $(i+1)^{\text{st}}$ blocks are T and $J - T$, respectively, with the remaining blocks composed of zero matrices. For this construction, we have:

$$X^* ' X^* = \begin{bmatrix} sN & \underline{N1}' & \underline{N1}' & \underline{N1}' & \dots & \underline{N1}' \\ \underline{N1} & A & B & 0 & & 0 \\ \underline{N1} & B & A & B & & 0 \\ \underline{N1} & 0 & B & A & & 0 \\ \vdots & & & & & \vdots \\ \underline{N1} & 0 & 0 & 0 & \dots & A \end{bmatrix},$$

where $A = T' T + (J-T)'(J-T)$ and $B = (J-T)' T$.

5. EXTENSION TO INCLUDE INTERACTION TERMS

If s is a prime or prime power, it is possible to include interactions in the $(0,1)$ representation of the s^n factorial. This representation is in terms of the geometric definition of the factorial effects. In this definition of the factorial effects the symbol $F_i F_j^a$ $a \neq 0 \in GF(s)$ is used to denote $(s-1)$ degrees of freedom belonging to the interaction of the i^{th} and j^{th} factors. As a ranges over the $s-1$ nonzero values of the Galois field of order s , $GF(s)$, all $(s-1)(s-1)$ degrees of freedom for the interaction between F_i and F_j are identified. A general k^{th} order component with $(s-1)$ degrees of freedom is denoted by $F_{i_1} F_{i_2}^{a_2} \dots F_{i_k}^{a_k}$ where a_2, a_3, \dots, a_k are nonzero elements of $GF(s)$. As the a_2, a_3, \dots, a_k range over all possible nonzero values, all $(s-1)^{k-1}(s-1)$ degrees of freedom associated with this k^{th} order interaction are identified.

If T denotes a design for the s^n factorial, let the columns of T be denoted as $T = [\underline{d}_1 \ \underline{d}_2 \ \dots \ \underline{d}_n]$. To include the interaction between factors F_i and F_j in the model, adjoin to T the $(s-1)$ columns

$$\underline{d}_i + a \underline{d}_j \quad a \neq 0 \in GF(s), \quad (5.1)$$

where all calculations are in the field $GF(s)$. Each of these columns clearly contains only the elements of $GF(s)$ and hence have the same form as the columns of T . For higher order interactions, say $F_{i_1} F_{i_2}^{a_2} \dots F_{i_k}^{a_k}$, we adjoin to T columns of the form.

$$\underline{d}_{i_1} + a_{i_2} \underline{d}_{i_2} + \dots + a_{i_k} \underline{d}_{i_k} \quad a_{i_1}, a_{i_2}, \dots, a_{i_k} \neq 0 \in GF(s) \quad (5.2)$$

Let D denote the $N \times (n+m)$ matrix with m columns adjoined to T for all desired interactions. The matrix D has elements from $GF(s)$ and as in (2.1), we let D_i denote the $N \times (n+m)$ incidence matrix of i in D , $i \in GF(s)$.

Then

$$\sum_{i \in GF(s)} D_i = J_{N \times (n+m)} \quad \text{and} \quad D = \sum_{i \in GF(s)} i D_i. \quad (5.3)$$

The X matrix for the model containing interaction terms has the same form as (2.3), that is

$$X = [1 : D_0 - D_1 : D_0 + D_1 - 2 D_2 : \dots : \sum_{i=0}^{s-2} D_i - (s-1) D_{s-1}] \quad (5.4)$$

It is now apparent that with $v = 1 + (n+m)(s-1)$, the $v \times v$ matrix G of equation (2.5) may be multiplied by $X_{N \times (n+m)}$ exactly as in (2.4) to produce a (0,1) representation of X. This observation is explicitly stated in Theorem 5.1.

Theorem 5.1. With X as in (5.4) and the transformations

$XG = X^*$, we have

$$(a) \quad X^* = [1 : D_1 : D_2 : \dots : D_{s-1}],$$

$$(b) \quad |G| = (s!)^{-(n+m)} (-1)^{(n+m)(s-1)},$$

$$(c) \quad |X| = X^* G^{-1}, \text{ and}$$

$$(d) \quad |X'X| = (s!)^{2(n+m)} |X^*'X^*|.$$

Proof. The theorem follows directly from theorem 2.1

It should be noted that in the asymmetric factorial Π_{s_i} that interactions between factors with the same (prime power) number of levels may be included in the model exactly as in the discussion above. For factors with differing numbers of levels or with non prime power number of levels the convenient field of order s does not exist. There may be a corresponding (0,1) representation which includes interaction terms for the general asymmetric factorial relative to some other formulation of the interaction contrasts.

Pesotan and Raktoe [1975] show that the (0,1) representation does not extend in a natural way if the product definition of the effects is used. They do show that such a representation does exist in terms of (-1,0,1) matrices, and exhibit suitable classes of design matrices T and sets of factorial effects such that a natural (0,1) representation does exist.

Example 5.1. Consider a case where there are three factors each at 3 levels and it is desired to include in the model the interaction of F_1 with each of F_2 and F_3 , but the F_2 by F_3 interaction and the three factor interaction are to be excluded. For any design T, we would thus adjoin the four columns.

$$\underline{d}_1 + \underline{d}_2, \underline{d}_1 + 2\underline{d}_2, \underline{d}_1 + \underline{d}_3, \text{ and } \underline{d}_1 + 2\underline{d}_3 \quad (\text{mod } 3).$$

The matrix D thus has seven columns and the corresponding X^* matrix of theorem 2.1 has $1 + 2(7) = 15$ columns.

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